

Tail generating functions for extendable branching processes

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Abstract

We study branching processes of independently splitting particles in the continuous time setting. If time is calibrated such that particles live on average one unit of time, the corresponding transition rates are fully determined by the generating function f for the offspring number of a single particle. We are interested in the defective case $f(1) = 1 - \epsilon$, where each splitting particle with probability ϵ is able to terminate the whole branching process. A branching process $\{Z_t\}_{t \geq 0}$ will be called *extendable* if $f(q) = q$ and $f(r) = r$ for some $0 \leq q < r < \infty$. Specializing on the extendable case we derive an integral equation for $F_t(s) = \mathbb{E}s^{Z_t}$. This equation is expressed in terms of what we call, *tail generating functions*. With help of this equation, we obtain limit theorems for the time to termination as $\epsilon \rightarrow 0$. We find that conditioned on non-extinction, the typical values of the termination time follow an exponential distribution in the nearly subcritical case, and require different scalings depending on whether the reproduction regime is asymptotically critical or supercritical. Using the tail generating function approach we also obtain new refined asymptotic results for the regular branching processes with $f(1) = 1$.

1 Introduction

We consider a single type Markov branching process $\{Z_t\}_{t \geq 0}$ with continuous time, assuming $Z_0 = 1$. This is a basic stochastic model for a population of particles having exponential life lengths with a parameter λ . It is thought that each particle at the moment of death is replaced by a random number of offspring particles according to a common reproduction law having a probability generating function

$$f(s) = \sum_{k=0}^{\infty} s^k p_k, \quad s \in [0, 1].$$

Without loss of generality, we will always assume that $\lambda = 1$. To recover a general λ case from our results, one should just replace the time variable t by λt . We also exclude the trivial case $p_1 = 1$.

Under the natural assumption $f(1) = 1$, the population mean formula $\mathbb{E}(Z_t) = e^{(m_1 - 1)t}$, where $m_1 = f'(1)$, identifies three different regimes of reproduction: subcritical, critical, and supercritical, depending on whether m_1 is smaller, equal, or larger than 1. The probability of ultimate extinction of the branching process, $q = \mathbb{P}(Z_\infty = 0)$, equals 1 in the subcritical and critical cases, and in the supercritical case it is given by the smallest non-negative root of the equation $f(x) = x$, see for example [5, 8].

In this paper we allow for defective probability distributions by letting $f(1) < 1$. In this case, each particle with probability $1 - f(1)$ sends the Markov process $\{Z_t\}$ to an absorbing graveyard state Δ . Such non-regular branching processes has got a limited attention in the literature. In [7], this setting for the linear birth-death processes was interpreted as a population model with killing. A related account on a special class of branching processes allowing for explosive particles is given in [13]. Another, biologically relevant interpretation for the termination event is favourable mutation. Think of a branching process governed by a subcritical reproduction law g with $g(1) = 1$ and $g'(1) < 1$, which may escape extinction due to a mutation switching the reproduction rate for the new type of particles into a supercritical regime [14]. Such a process stopped at the first mutation event, can be modelled by a single type branching process

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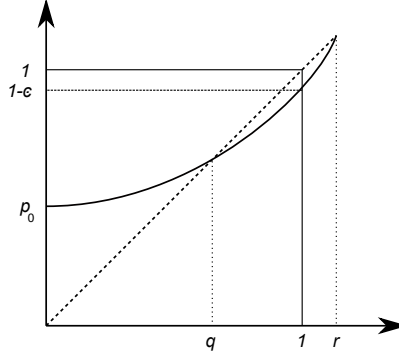


Figure 1: A (q, r) -extendable probability generating function $f(s)$.

with $f(s) = g(s(1 - \mu))$, where μ is the probability for a particle to mutate at birth. If μ is small, then $1 - f(1) = 1 - g(1 - \mu) \sim \mu g'(1)$.

Another non-regular case, not addressed here, is that of explosive branching processes with $f(1) = 1$. The interested reader is referred to [12] investigating a broad class of such non-regular Markov processes.

Definition 1 For $0 \leq q \leq 1 \leq r < \infty$ and $q < r$, we say that a possibly defective probability generating function f is (q, r) -extendable, if $f(q) = q$ and $f(r) = r$. A branching process whose reproduction law has a (q, r) -extendable generating function will be called a (q, r) -extendable branching process.

Figure 1 depicts a graph for a (q, r) -extendable probability generating function with $f(1) < 1$. The focus of this paper is on the (q, r) -extendable branching processes. In particular, when $f(1) = 1$, our results apply to the extendable subcritical case with $q = 1 < r < \infty$ and $0 < m_1 < 1 < f'(r)$, as well as to the supercritical case with $0 \leq q < 1 = r$ and $0 \leq f'(q) < 1 < m_1 < \infty$. The subcritical extendable branching processes arise naturally as supercritical branching processes conditioned on extinction, see [4] and [6].

Theorem 3 below, proposes a new form of the backward Kolmogorov equation valid for the (q, r) -extendable branching processes. It is expressed in terms of what we call *tail generating functions*. The name comes from a simple observation involving the tail probabilities of the reproduction law:

$$\frac{f(1) - f(s)}{1 - s} = \sum_{i \geq 0} s^i \sum_{j \geq i+1} p_j.$$

The last generating function will be denoted $f^{(2)}(1, s)$ for $s \in [0, 1)$, and also by continuity, we will put $f^{(2)}(1, 1) = f'(1) = m_1$.

Definition 2 Given a power series $v(s) = \sum_{k=0}^{\infty} s^k v_k$ with all $v_k \geq 0$, define its n -th order tail generating function by

$$v^{(n)}(s_1, \dots, s_n) = \sum_{i_1 \geq 0, \dots, i_n \geq 0} v_{j_n} s_1^{i_1} \dots s_n^{i_n},$$

where $j_n = i_1 + \dots + i_n + n - 1$, $n \geq 1$.

Observe that the tail generating functions are symmetric functions which can be computed using a simple recursion

$$v^{(n)}(s_1, \dots, s_n) = \frac{v^{(n-1)}(s_1, \dots, s_{n-1}) - v^{(n-1)}(s_2, \dots, s_n)}{s_1 - s_n}, \quad s_1 \neq s_n. \quad (1)$$

Whenever some of the arguments coincide, the following rule applies, see Section 6,

$$v^{(k+n)}(s_1, \dots, s_{k-1}, s, \dots, s) = \frac{1}{n!} \frac{d^n}{ds^n} v^{(k)}(s_1, \dots, s_{k-1}, s), \quad k \geq 1, n \geq 1. \quad (2)$$

In particular, the second order tail generating functions satisfy

$$v^{(2)}(s_1, s_2) = \frac{v(s_1) - v(s_2)}{s_1 - s_2}, \quad s_1 \neq s_2, \quad v^{(2)}(s, s) = v'(s),$$

so that for a given (q, r) -extendable probability generating function f , we have

$$f^{(2)}(q, s) = \frac{f(s) - q}{s - q}, \quad f^{(2)}(q, q) = f'(q), \quad f^{(2)}(q, r) = 1, \quad f^{(2)}(r, s) = \frac{r - f(s)}{r - s}, \quad f^{(2)}(r, r) = f'(r).$$

Theorem 3 *Let $\{Z_t\}$ be a (q, r) -extendable branching process. Then for $t > 0$, the probability generating function $F_t(s) = E(s^{Z_t})$ is (q, r) -extendable. If $f'(r) < \infty$, then*

$$F_t'(q) = e^{-\alpha t}, \quad \alpha = 1 - f'(q), \quad F_t'(r) = e^{\beta t}, \quad \beta = f'(r) - 1,$$

$\gamma = \alpha/\beta \in (0, 1]$, and for $s \in [0, q) \cup (q, r)$,

$$\frac{F_t(s) - q}{s - q} = e^{-\alpha t} \left\{ \frac{r - F_t(s)}{r - s} \right\}^\gamma \exp \left\{ \int_s^{F_t(s)} \psi_{q,r}(x) dx \right\}, \quad (3)$$

where

$$\psi_{q,r}(x) = \frac{f^{(4)}(q, q, r, x) - \gamma f^{(4)}(q, r, r, x)}{f^{(3)}(q, r, x)}. \quad (4)$$

The integrand $\psi_{q,r}(x)$ appearing in Theorem 3 has no singularities over the interval $x \in [0, r)$, and

$$\psi_{q,r}(q) = \frac{\gamma}{r - q} - \frac{f''(q)}{2}, \quad \psi_{q,r}(r) = \frac{1}{r - q} - \frac{\gamma f''(r)}{2\beta}. \quad (5)$$

Theorem 3 is proved in Section 2. Section 3 contains Theorem 7 addressing the critical case, which corresponds to the parameter option $(q, r) = (1, 1)$ excluded from Theorem 3. Theorem 7 proposes equation (12) as a counterpart of (3) for the critical branching processes. Section 4 discusses an important special case of equations (3) and (12) where the integral parts vanish due to $\psi_{q,r}(x) \equiv 0$. Condition $\psi_{q,r}(x) \equiv 0$ leads to a four-parameter family of possibly defective probability distributions. These, what we call, *modified linear-fractional distributions*, have interest of its own as an extension of the well-known family of linear-fractional distributions. Another illuminating case, where the integrals in equations (3) and (12) are computed explicitly, is presented in Section 5.

In Section 6 we analyse finiteness of the integrals $\int_0^r \psi_{q,r}(x) dx$ connected to Theorems 3 and 7. We find that the $x \log x$ -type conditions playing a crucial role in the theory of branching processes [1], are expressed naturally in terms of tail generating functions. In Section 7 we apply the tail generating function approach to obtain a novel Yaglom type theorem for extendable branching processes conditioned on $0 < Z_t < \infty$.

Note that branching processes with $f(1) < 1$ fall outside the usual classification system, as irrespective of the value of m_1 , the probability of ultimate extinction q is always less than one, see Section 2. In Section 8 we consider a family of (q_ϵ, r_ϵ) -extendable branching processes such that for some $s_0 > 1$,

$$f_\epsilon(s) \rightarrow f(s), \quad f(1) = 1, \quad f(s_0) < \infty, \quad f(s_0) \neq s_0, \quad s \in [0, s_0], \quad (6)$$

as $\epsilon \rightarrow 0$. In this setting we can speak of nearly subcritical, critical, and supercritical extendable branching processes. We study the distribution of the termination time conditioned on non-extinction, and conclude that the largest values of the termination time (proportional to $1/\sqrt{\epsilon}$) are expected in the balanced nearly critical case.

Finally, in Sections 9 - 10 we apply the tail generating function approach to the regular case $f(1) = 1$. Using Theorems 3, 7, and results from Section 6 we obtain a new refined asymptotic formula for critical branching processes, and then give streamlined proofs for the known facts in the supercritical case.

2 Proof of Theorem 3

Lemma 4 Consider a branching process with $f(1) \leq 1$. Its probability of extinction $q = P(Z_\infty = 0)$ is the smallest non-negative root of $f(x) = x$. The (possibly defective) probability generating functions $x_t = F_t(s)$ of the branching process satisfy the backward Kolmogorov equation

$$\frac{dx_t}{dt} = f(x_t) - x_t, \quad x_0 = s, \quad s \in [0, 1). \quad (7)$$

PROOF Let L and X be the life length and offspring number of the ancestral particle. In the defective case with $f(1) < 1$ we assume that $1 - f(1) = P(X = \Delta)$ and $s^\Delta = 0$. Then by the branching property,

$$Z_t = 1_{\{L > t\}} + 1_{\{L \leq t\}} 1_{\{X \neq \Delta\}} \sum_{i=1}^X Z_{t-L}^{(i)} + 1_{\{L \leq t\}} 1_{\{X = \Delta\}} \cdot \Delta,$$

where $Z_u^{(i)}$ stands for the branching process stemming from the i -th ancestral daughter. This yields in term of generating functions

$$F_t(s) = se^{-t} + \int_0^t f(F_{t-u}(s))e^{-u} du,$$

due to the assumption of exponential life length and independence among daughter particles. Multiplying by e^t and taking the derivatives we derive the ordinary differential equation (7).

Turning to the probability of extinction $Q := P(Z_\infty = 0)$, observe that since

$$Q = E(P(Z_\infty = 0 | Z_t)) = E(Q^{Z_t}) = F_t(Q), \quad t \geq 0,$$

equation (7) entails that Q is a root of $f(x) = x$. This also gives the smallest non-negative root, because

$$P(Z_t = 0) = F_t(0) \nearrow Q, \quad t \rightarrow \infty.$$

■

Lemma 5 Consider a (q, r) -extendable branching process. Then for $t \geq 0$, we have $F_t(q) = q$, $F_t(r) = r$, and

$$t = \int_s^{F_t(s)} \frac{dx}{f(x) - x}, \quad s \in [0, q) \cup (q, r). \quad (8)$$

PROOF The integral equation (8) follows from the backward Kolmogorov equation (7). The singularity point $x = q$ of the integrand is circumvented as each solution of (7) with $x_0 \in [0, q)$ is such that $x_t \in [0, q)$ for all $t \geq 0$, while each solution of (7) with $x_0 \in (q, r)$ is such that $x_t \in (q, r)$ for all $t \geq 0$. Moreover, (7) implies that $F_t(q) = q$ and $F_t(r) = r$, so that for each $t > 0$, the probability generating function $F_t(s)$ is (q, r) -extendable. ■

Equation (3) is obtained from (8) by extracting principal terms associated with the singularity points q and r of the integrand. We compute these terms with help of the following lemma.

Lemma 6 For a given (q, r) -extendable generating function f define $\phi(s) = f^{(3)}(q, r, s)$. Then

$$\begin{aligned} \phi(s) &= \frac{f(s) - s}{(q - s)(r - s)}, \quad s \in [0, r), \\ \phi(q) &= f^{(3)}(q, q, r) = \frac{\alpha}{r - q}, \quad \phi'(q) = f^{(4)}(q, q, q, r) = \frac{\alpha}{(r - q)^2} - \frac{f''(q)}{2(r - q)}, \end{aligned}$$

where $\alpha = 1 - f'(q)$. Furthermore, if $f'(r) < \infty$, then

$$\phi(r) = f^{(3)}(q, r, r) = \frac{\beta}{r - q}, \quad \phi^{(2)}(q, r) = f^{(4)}(q, q, r, r) = \frac{\beta - \alpha}{(r - q)^2},$$

where $\beta = f'(r) - 1$. Finally, if $f''(r) < \infty$, then

$$\phi'(r) = f^{(4)}(q, r, r, r) = \frac{f''(r)}{2(r - q)} - \frac{\beta}{(r - q)^2}.$$

PROOF The first stated equality follows from the definition of ϕ , q , and r . Observe that by monotonicity of ϕ , we have $0 < \alpha \leq \beta \leq \infty$, where $\alpha = \beta$ holds if and only if $\phi(s) \equiv p_2$ is a constant, that is when the possible numbers of offspring are 0, 1, or 2:

$$f(s) = s + (s - q)(s - r)p_2 = qrp_2 + (1 - qp_2 - rp_2)s + p_2s^2.$$

This yields one of the statements of Theorem 3 claiming that $\gamma = \alpha/\beta$ belongs to $(0, 1]$.

The other stated formulas are obtained using (1) and (2). For example, the last statement is valid since

$$f^{(4)}(q, r, r, r) = \frac{f^{(3)}(r, r, r) - f^{(3)}(q, r, r)}{r - q} = \frac{f''(r)}{2(r - q)} - \frac{f^{(2)}(r, r) - f^{(2)}(q, r)}{(r - q)^2} = \frac{f''(r)}{2(r - q)} - \frac{f'(r) - 1}{(r - q)^2}.$$

■

PROOF OF THEOREM 3. Consider a (q, r) -extendable branching process. Provided $f'(r) < \infty$, we have

$$\frac{\phi(r)}{(q - x)(r - x)\phi(x)} = \frac{1}{(q - x)(r - x)} + \frac{\phi^{(2)}(r, x)}{(q - x)\phi(x)} = \frac{\phi^{(2)}(r, x)(r - q) + \phi(x)}{(r - q)(q - x)\phi(x)} - \frac{1}{(r - q)(r - x)},$$

implying

$$\frac{1}{(q - x)(r - x)\phi(x)} = \frac{\phi^{(2)}(r, x)(r - q) + \phi(x)}{\beta(q - x)\phi(x)} - \frac{1}{\beta(r - x)}.$$

By (8) and Lemma 6,

$$t = \int_s^{F_t(s)} \frac{dx}{(q - x)(r - x)\phi(x)},$$

and it follows

$$t = \frac{1}{\beta} \ln F_t^{(2)}(r, s) - \int_s^{F_t(s)} \frac{\phi^{(2)}(r, x)dx}{\beta\phi(x)} + \int_s^{F_t(s)} \frac{\phi(r)dx}{\beta(q - x)\phi(x)}, \quad (9)$$

where $F_t^{(2)}(s_1, s_2)$ is the second order tail generating function for $F_t(s)$. The last integral equals

$$\begin{aligned} \int_s^{F_t(s)} \frac{dx}{(r - q)(q - x)\phi(x)} &= \int_s^{F_t(s)} \frac{dx}{\alpha(q - x)} + \int_s^{F_t(s)} \frac{(\phi(q) - \phi(x))dx}{\alpha(q - x)\phi(x)} \\ &= -\frac{\ln F_t^{(2)}(q, s)}{\alpha} + \int_s^{F_t(s)} \frac{\phi^{(2)}(q, x)dx}{\alpha\phi(x)}, \end{aligned}$$

and we conclude that for $s \in [0, r)$,

$$t = \frac{\ln F_t^{(2)}(r, s)}{\beta} - \frac{\ln F_t^{(2)}(q, s)}{\alpha} + \int_s^{F_t(s)} \frac{\phi^{(2)}(q, x)dx}{\alpha\phi(x)} - \int_s^{F_t(s)} \frac{\phi^{(2)}(r, x)dx}{\beta\phi(x)}, \quad (10)$$

which is equivalent to (3), since $\gamma = \alpha/\beta$ and

$$\psi_{q,r}(x) = \frac{\phi^{(2)}(q, x)}{\phi(x)} - \frac{\gamma\phi^{(2)}(r, x)}{\phi(x)}.$$

■

3 Tail generating functions for critical branching processes

In this section we assume $f(1) = 1$ and $m_1 = f'(1) = 1$. Denote $f_k(x) = f^{(k)}(1, \dots, 1, x)$ for $k \geq 2$, so that for $x \in [0, 1)$,

$$f_2(x) = \frac{1 - f(x)}{1 - x}, \quad f_3(x) = \frac{f(x) - x}{(1 - x)^2}, \quad f_4(x) = \frac{m_2 - f_3(x)}{1 - x}, \quad f_5(x) = \frac{m_3 - f_4(x)}{1 - x}, \quad (11)$$

where

$$m_2 = f_3(1) = \frac{f''(1)}{2}, \quad m_3 = f_4(1) = \frac{f'''(1)}{6}.$$

Notice that in the critical case parameters m_2 and m_3 are directly related to the centered moments of the reproduction law due to

$$\begin{aligned} f''(1) &= \sum_{k \geq 2} k(k-1)p_k = \sum_{k \geq 0} ((k-1)^2 + k-1)p_k = \sum_{k \geq 0} (k-1)^2 p_k, \\ f'''(1) &= \sum_{k \geq 3} k(k-1)(k-2)p_k = \sum_{k \geq 0} ((k-1)^2 - 1)(k-1)p_k = \sum_{k \geq 0} (k-1)^3 p_k. \end{aligned}$$

Theorem 7 *If $f(1) = 1$, $m_1 = 1$, and $f'''(1) < \infty$, then for $t \geq 0$ and $s \in [0, 1)$,*

$$t = \frac{F_t(s) - s}{m_2(1-s)(1-F_t(s))} - \frac{m_3}{m_2^2} \ln \frac{1 - F_t(s)}{1 - s} + \int_s^{F_t(s)} \psi_{1,1}(x) dx, \quad (12)$$

where

$$\psi_{1,1}(x) = \frac{f_4^2(x)}{m_2^2 f_3(x)} - \frac{f_5(x)}{m_2^2}. \quad (13)$$

PROOF Using (11) and Lemma 8 we derive

$$t = \int_s^{F_t(s)} \frac{dx}{(1-x)^2 f_3(x)} = \int_s^{F_t(s)} \frac{dx}{(1-x)^2 m_2} + \int_s^{F_t(s)} \frac{f_4(x) dx}{(1-x) m_2 f_3(x)}.$$

This gives

$$m_2 t = \frac{1}{1 - F_t(s)} - \frac{1}{1 - s} + \int_s^{F_t(s)} \frac{f_4(x) dx}{(1-x) f_3(x)},$$

which together with

$$\frac{f_4(x)}{(1-x) f_3(x)} - \frac{f_4(x)}{(1-x) m_2} = \frac{f_4^2(x)}{m_2 f_3(x)}$$

yield

$$m_2^2 t = \frac{m_2(F_t(s) - s)}{(1-s)(1-F_t(s))} + \int_s^{F_t(s)} \frac{f_4(x) dx}{1-x} + \int_s^{F_t(s)} \frac{f_4^2(x) dx}{f_3(x)}.$$

Now, to deduce (12), it remains to use equalities

$$\frac{f_4(x)}{1-x} = \frac{m_3}{1-x} - f_5(x), \quad \int_s^{F_t(s)} \frac{dx}{1-x} = \ln \frac{1 - F_t(s)}{1 - s}.$$

■

We will show next that the critical case equation (12) is linked to the non-critical case equation (10) via a continuity argument, although the components of these two equations look different. For this we need the next observation.

Lemma 8 Consider a (q, r) -extendable branching process with $f'(r) < \infty$, then for every $t \geq 0$ and $x \in [0, r)$,

$$\frac{\psi_{q,r}(x)}{\alpha} = \frac{1}{f^{(3)}(q, q, r)f^{(3)}(q, r, r)} \left\{ \frac{f^{(4)}(q, q, r, x)f^{(4)}(q, r, r, x)}{f^{(3)}(q, r, x)} - f^{(5)}(q, q, r, r, x) \right\}, \quad (14)$$

and furthermore,

$$\frac{\ln F_t^{(2)}(r, s)}{\beta} - \frac{\ln F_t^{(2)}(q, s)}{\alpha} = \frac{1}{f^{(3)}(r, r, q)} \frac{F_t^{(3)}(q, r, s)}{F_t^{(2)}(q, s)} e_{q,r}(t, s) - \frac{f^{(4)}(q, q, r, r)}{f^{(3)}(q, q, r)f^{(3)}(q, r, r)} \ln F_t^{(2)}(q, s), \quad (15)$$

where $F_t^{(n)}(s_1, \dots, s_n)$ is the n -th order tail generating function for $F_t(s)$ and

$$e_{q,r}(t, s) = e^{\left\{ (r-q) \frac{F_t^{(3)}(q, r, s)}{F_t^{(2)}(q, s)} \right\}}, \quad e\{x\} = \frac{\ln(1+x)}{x}.$$

PROOF Recall the definition of ϕ and its properties obtained in Lemma 6. By a telescopic rearrangement,

$$\begin{aligned} \frac{\phi^{(2)}(q, x)}{\alpha\phi(x)} - \frac{\phi^{(2)}(r, x)}{\beta\phi(x)} &= \frac{\phi^{(2)}(q, x)}{\alpha\phi(x)} - \frac{\phi^{(2)}(q, x)}{\alpha\phi(r)} + \frac{\phi^{(2)}(q, x)}{\alpha\phi(r)} - \frac{\phi^{(2)}(r, x)}{\beta\phi(q)} + \frac{\phi^{(2)}(r, x)}{\beta\phi(q)} - \frac{\phi^{(2)}(r, x)}{\beta\phi(x)} \\ &= \frac{(r-q)(r-x)\phi^{(2)}(r, x)\phi^{(2)}(q, x)}{\alpha\beta\phi(x)} + \frac{(r-q)(\phi^{(2)}(q, x) - \phi^{(2)}(r, x))}{\alpha\beta} \\ &\quad + \frac{(r-q)(x-q)\phi^{(2)}(r, x)\phi^{(2)}(q, x)}{\alpha\beta\phi(x)} = \frac{(r-q)^2}{\alpha\beta} \left\{ \frac{\phi^{(2)}(q, x)\phi^{(2)}(r, x)}{\phi(x)} - \phi^{(3)}(q, r, x) \right\}, \end{aligned}$$

which entails (14). On the other hand, in view of

$$\frac{F_t^{(2)}(r, s)}{F_t^{(2)}(q, s)} = 1 + \frac{(r-q)F_t^{(3)}(q, r, s)}{F_t^{(2)}(q, s)},$$

we obtain

$$\frac{\ln F_t^{(2)}(r, s)}{\beta} - \frac{\ln F_t^{(2)}(q, s)}{\alpha} = \frac{1}{\beta} \ln \left\{ 1 + \frac{(r-q)F_t^{(3)}(q, r, s)}{F_t^{(2)}(q, s)} \right\} - \frac{(\beta-\alpha) \ln F_t^{(2)}(q, s)}{\alpha\beta}.$$

It remains to see that the right hand side equals to that of (15). ■

Consider a family of (q_ϵ, r_ϵ) -extendable branching processes satisfying (6) and denote by $F_{t,\epsilon}(s)$ their probability generating functions. If $f'(1) = 1$, then the function $F_{t,\epsilon}(s)$ and its limit $F_t(s)$ satisfy equations (10) and (12) respectively. Applying Lemma 8 we see that there is a term by term agreement between (10) and (12). Indeed, by (14) we can write $\alpha_\epsilon^{-1}\psi_{q_\epsilon, r_\epsilon}(x) \rightarrow \psi_{1,1}(x)$. On the other hand, $e_{q_\epsilon, r_\epsilon}(t, s) \rightarrow 1$ as $r_\epsilon - q_\epsilon \rightarrow 0$, so that (15) eventually implies

$$\frac{\ln F_{t,\epsilon}^{(2)}(r_\epsilon, s)}{\beta_\epsilon} - \frac{\ln F_{t,\epsilon}^{(2)}(q_\epsilon, s)}{\alpha_\epsilon} \rightarrow \frac{1}{m_2} \frac{F_t^{(3)}(1, 1, s)}{F_t^{(2)}(1, s)} - \frac{m_3}{m_2^2} \ln F_t^{(2)}(1, s).$$

4 Modified linear-fractional reproduction law

Definition 9 A (possibly defective) probability distribution will be called *modified linear-fractional*, if its generating function has the form

$$f(s) = p_0 + p_1 s + (1 - p_0 - p_1 - p_\Delta) s^2 (1-p)(1-ps)^{-1}, \quad s \in [0, p^{-1}), \quad (16)$$

for some combination of four parameters (p_0, p_1, p_Δ, p) satisfying

$$p_0 \in [0, 1), \quad p_1 \in [0, 1), \quad p_\Delta \in [0, 1), \quad p_0 + p_1 + p_\Delta < 1, \quad p \in [0, 1).$$

A random variable X having a modified linear-fractional distribution is characterised by the following shifted geometric property

$$P(X = 2 + k | 2 \leq X < \infty) = (1 - p)p^k, \quad k \geq 0.$$

Definition 9 is a generalisation of the well-known linear-fractional (or zero-modified geometric) distribution with

$$f(s) = p_0 + (1 - p_0)s(1 - p)(1 - ps)^{-1}.$$

Indeed, putting $p_1 = (1 - p_0 - p_\Delta)(1 - p)$ into (16) we arrive at a possibly defective linear-fractional generating function

$$f(s) = p_0 + (1 - p_0 - p_\Delta)s(1 - p)(1 - ps)^{-1}, \quad (17)$$

with $p_0, p_\Delta, p \in [0, 1)$ and $p_0 + p_\Delta < 1$.

Lemma 10 *Consider a modified linear-fractional f given by Definition 9. If*

$$p_\Delta = 0, \quad p_0 \in (0, 1), \quad p_1 = 1 - p_0(2 - p) \in [0, 1), \quad p \in [0, 1),$$

then

$$f(1) = f'(1) = 1, \quad m_2 = \frac{p_0}{1 - p}, \quad m_3 = \frac{p_0 p}{(1 - p)^2},$$

and

$$f(s) = s + p_0(1 - s)^2(1 - ps)^{-1}, \quad p_0 \in (0, 1), \quad p \in [0, 1). \quad (18)$$

In all other cases f is a (q, r) -extendable probability generating function such that

$$f(s) = s + \frac{\alpha(q - s)(r - s)}{r - q\gamma - (1 - \gamma)s}, \quad s \in [0, r), \quad (19)$$

where besides the usual conditions on (q, r, α, γ) :

$$0 \leq q \leq 1 \leq r < \infty, \quad q < r, \quad \alpha \in (0, 1), \quad \gamma \in (0, 1], \quad (20)$$

the following extra restriction holds

$$\alpha \leq 1 - \gamma + \frac{(r - q)^2 \gamma}{r^2 - \gamma q^2}. \quad (21)$$

PROOF Clearly, in the modified linear-fractional distribution case

$$m_1 = p_1 + (1 - p_0 - p_1 - p_\Delta) \frac{2 - p}{1 - p},$$

so that given $f(1) = 1$ we get

$$m_1 = 1 + \frac{1 - p_0(2 - p) - p_1}{1 - p}.$$

Having this in mind, the critical case formula (18) is easily checked. Observe also that in the critical case

$$f_{2+k}(s) = f^{(2+k)}(1, \dots, 1, s) = \frac{p_0 p^k}{(1 - p)^k (1 - ps)}, \quad k \geq 0.$$

In the non-critical case, taking into account that q and r are roots of equation $f(x) = x$, relation (16) can be rewritten as

$$f(s) = s + \frac{(q - s)(r - s)c}{1 - ps},$$

with some $c \in (0, \infty)$. This means $\phi(s) = \frac{c}{1-ps}$, so that in view of $\phi(q) = \frac{\alpha}{1-pq}$ and $\phi(r) = \frac{\beta}{1-pr}$ we obtain (19). Comparing (19) with (16), we find

$$p_0 = \frac{\alpha qr}{r - \gamma q}, \quad p_1 = 1 - \frac{\alpha(r^2 - \gamma q^2)}{(r - \gamma q)^2}, \quad p_\Delta = \frac{\alpha(r-1)(1-q)}{r-1+\gamma(1-q)}, \quad p = \frac{1-\gamma}{r-\gamma q}. \quad (22)$$

These relations imply that the conditions $p_0 \geq 0$, $p_1 < 1$, $p_\Delta \geq 0$, $0 \leq p < 1$ are always satisfied. Restriction (21) stems from $p_1 \geq 0$. No further restrictions are needed, since

$$\begin{aligned} 1 - p_0 - p_1 - p_\Delta &= \frac{\alpha(r^2 - \gamma q^2)}{(r - \gamma q)^2} - \frac{\alpha qr}{r - \gamma q} - \frac{\alpha(r-1)(1-q)}{r-1+\gamma(1-q)} \\ &= \frac{\alpha[(r^2(1-q) + \gamma q^2(r-1))(r-1+\gamma(1-q)) - (r-1)(1-q)(r-\gamma q)^2]}{(r - \gamma q)^2(r-1+\gamma(1-q))} \\ &= \frac{\alpha\gamma(r-q)^2}{(r - \gamma q)^2(r-1+\gamma(1-q))} \end{aligned}$$

implies $p_0 + p_1 + p_\Delta < 1$. ■

Proposition 11 *For the functions defined by (4) and (13), condition $\psi_{q,r}(x) \equiv 0$ holds if and only if f has the form (16). Given (16), the probability generating function $F_t(s)$ of the corresponding branching process satisfies in the non-critical case*

$$\frac{F_t(s) - q}{s - q} = e^{-\alpha t} \left\{ \frac{r - F_t(s)}{r - s} \right\}^\gamma, \quad (23)$$

and in the critical case, with f given by (18),

$$p_0 t = (1-p) \frac{F_t(s) - s}{(1-s)(1-F_t(s))} - p \ln \frac{1 - F_t(s)}{1 - s}. \quad (24)$$

PROOF To prove the stated criterium, observe that in terms of Lemma 6, condition $\psi_{q,r}(x) \equiv 0$ is equivalent to

$$\beta\phi^{(2)}(q, s) = \alpha\phi^{(2)}(r, s), \quad s \in [0, r]. \quad (25)$$

Using relations from Lemma 6 we see that the last relation is equivalent to (19), which in turn is equivalent to (16) by Lemma 10. Equations (23) and (24) directly follow from Theorems 3 and 7. ■

Remark. According to (22) with the special choice of $(q, r) = (0, 1)$, the modified linear-fractional probability generating function (16) becomes

$$h(s) = (1-\alpha)s + \frac{\alpha\gamma s^2}{1 - (1-\gamma)s}.$$

On the other hand, for any f given by (16), we have

$$f^{(2)}(q, q + (r-q)s) = \frac{\gamma s + (1-\alpha)(1-s)}{1 - (1-\gamma)s} = \frac{h(s)}{s}.$$

This implies a representation

$$f(s) = q + (r-q)h\left(\frac{s-q}{r-q}\right)$$

that can be interpreted in the following way. We can treat the pair of fixed points (q, r) as scaling parameters, and the pair (α, γ) as shape parameters for the family of modified linear-fractional distributions. Recall that parametrisation (q, r, α, γ) is subject to restrictions (20) and (21).

Notice also, that a linear-fractional generating function (17) is fully defined by a triplet (q, r, γ) which corresponds to (q, r, α, γ) with $\alpha = 1 - \gamma$. In this case restriction (21) is fulfilled automatically.

5 Reproduction with trifurcations

Putting $p_1 = p = 0$ into (16), we get a (possibly defective) binary splitting reproduction law

$$f(s) = p_0 + p_2 s^2, \quad p_0 + p_2 \leq 1.$$

The corresponding branching process is the linear birth-death process with killing studied in [7]. In this case equation (23) holds with $\gamma = 1$ and

$$\alpha = \sqrt{1 - 4p_0 p_2}, \quad q = \frac{1 - \alpha}{2p_2}, \quad r = \frac{1 + \alpha}{2p_2}.$$

It brings the well-known explicit linear-fractional solution for the non-critical case

$$F_t(s) = \frac{(r - s)q + (s - q)re^{-\alpha t}}{r - s + (s - q)e^{-\alpha t}}.$$

In the critical case, $p_0 = p_2 = \frac{1}{2}$, equation (24) becomes

$$1 - \frac{1 - F_t(s)}{1 - s} = \frac{t}{2} (1 - F_t(s)),$$

which yields the linear-fractional formula for the critical birth-death process

$$F_t(s) = 1 - \frac{1 - s}{1 + \frac{t}{2}(1 - s)}.$$

Further examples of explicit formulas for $F_t(s)$, going beyond the linear-fractional case, are presented in [13].

A less trivial example arises when *trifurcations* are also allowed. Consider a three-parameter family

$$f(s) = p_0 + p_2 s^2 + p_3 s^3, \quad p_3 > 0, \quad p_0 + p_2 + p_3 \leq 1.$$

Denote by (q, r, x_3) the roots of the third order algebraic equation $f(x) = x$: two non-negative roots $q \leq r$ and a negative solution x_3 . Then we can write

$$f(s) - s = p_3(s - q)(s - r)(s - x_3) = (s - q)(s - r)(p_3 s + w),$$

where $w = -p_3 x_3 \in (0, \infty)$. From

$$f(1) = 1 - (1 - q)(r - 1)(p_3 + w),$$

it is clear that $q \leq 1 \leq r$. Since $f'(0) = p_1 = 0$, and

$$f'(s) = 1 + (s - r)(p_3 s + w) + (s - q)(p_3 s + w) + p_3(s - q)(s - r),$$

we conclude that $w = \frac{1 + p_3 q r}{q + r}$.

Proposition 12 *Consider a branching process with the reproduction law*

$$f(s) = s + (s - q)(s - r)(p_3 s + w), \quad w = \frac{1 + p_3 q r}{q + r}.$$

If $q < r$, then

$$\frac{F_t(s) - q}{s - q} = e^{-\alpha t} \left\{ \frac{r - F_t(s)}{r - s} \right\}^\gamma \left(\frac{p_3 F_t(s) + w}{p_3 s + w} \right)^{1-\gamma}, \quad \gamma = \frac{p_3 q + w}{p_3 r + w}. \quad (26)$$

If $q = r = 1$, then

$$t = \frac{2}{1 + 3p_3} \frac{F_t(s) - s}{(1 - s)(1 - F_t(s))} - \frac{4p_3}{(1 + 3p_3)^2} \ln \frac{1 - F_t(s)}{1 - s} + \frac{4p_3}{(1 + 3p_3)^2} \ln \frac{1 + p_3 + 2p_3 F_t(s)}{1 + p_3 + 2p_3 s}.$$

PROOF After computing $f^{(3)}(q, r, s) = p_3s + w$, we find

$$\alpha = (r - q)(p_3q + w), \quad \beta = (r - q)(p_3r + w), \quad f^{(4)}(q, q, r, s) \equiv f^{(4)}(q, r, r, s) \equiv p_3,$$

so that given $q < r$, the function defined by (4) is computed explicitly

$$\psi_{q,r}(x) = \frac{(r - q)p_3^2}{(p_3r + w)(p_3x + w)}, \quad \int_{s_1}^{s_2} \psi_{q,r}(x)dx = (1 - \gamma) \ln \frac{p_3s_2 + w}{p_3s_1 + w}.$$

As a result, equation (3) simplifies and takes the form stated by the lemma.

In the critical case when $q = r = 1$, we get

$$f(s) = s + \frac{1}{2}(1 - s)^2(2p_3s + 1 + p_3), \quad f_3(s) = p_3s + \frac{1 + p_3}{2}, \quad f_4(s) = p_3, \quad f_5(s) = 0,$$

implying $m_2 = \frac{1+3p_3}{2}$ and $m_3 = p_3$. It follows that

$$\psi_{1,1}(x) = \frac{2p_3^2}{m_2^2(1 + p_3 + 2p_3x)}, \quad \int_{s_1}^{s_2} \psi_{1,1}(x)dx = \frac{4p_3}{(1 + 3p_3)^2} \ln \frac{1 + p_3 + 2p_3s_2}{1 + p_3 + 2p_3s_1},$$

and (12) implies the second stated equation. ■

Remark. If $f(s) = s^3$, then $q = 0$, $r = p_3 = w = 1$, and $\gamma = 1/2$, so that equation (26) takes the form

$$\frac{e^{2t}F_t^2(s)}{s^2} = \frac{1 - F_t^2(s)}{1 - s^2},$$

which can be solved explicitly. This is a particular case of the Harris-Yule process characterised by $f(s) = s^{k+1}$ for some $k \geq 1$. In this case an explicit expression is available:

$$F_t(s) = \left(e^{kt}s^{-k} - e^{kt} + 1 \right)^{-1/k}.$$

6 Tail generating functions and $x \log x$ -conditions

In this section we establish Theorem 13, which presents a criterium for a generalised $x \log x$ condition in terms of the tail generating functions. Using Theorem 13 we prove Propositions 14 and 15 addressing condition $\int_0^r |\psi_{q,r}(x)|dx < \infty$ for the functions (4) and (13).

We start by showing that the earlier announced relation (2) holds. Indeed, turning to Definition 2, we find

$$\begin{aligned} v^{(k+n)}(s_1, \dots, s_{k-1}, s, \dots, s) &= \sum_{i_1 \geq 0, \dots, i_n \geq 0} v_{j_k+n} s_1^{i_1} \dots s_{k-1}^{i_{k-1}} s^{i_k + \dots + i_{k+n}} \\ &= \sum_{i_1 \geq 0, \dots, i_k \geq 0} \binom{n + i_k}{n} v_{n+j_k} s_1^{i_1} \dots s_{k-1}^{i_{k-1}} s^{i_k} = \frac{1}{n!} \frac{d^n}{ds^n} v^{(k)}(s_1, \dots, s_{k-1}, s), \end{aligned}$$

where $j_k = i_1 + \dots + i_k + k - 1$. In particular,

$$v^{(n+2)}(a, \dots, a, s) = \sum_{i \geq 0} s^i \sum_{j \geq n} a^{j-n} \binom{j}{n} v_{i+j+1}, \quad n \geq 0. \quad (27)$$

Theorem 13 Let $f(s) = \sum_{k=0}^{\infty} s^k p_k$ be a (possibly defective) probability generating function, $a > 0$, and $n \geq 0$ be a non-negative integer. Then, the moment condition

$$\sum_{k=2}^{\infty} p_k a^k k^n \ln k < \infty \quad (28)$$

is equivalent to

$$\int_0^a f^{(n+2)}(a, \dots, a, x) dx < \infty.$$

PROOF Applying (27) we find

$$\begin{aligned} \int_0^a f^{(n+2)}(a, \dots, a, x) dx &= \sum_{j=0}^{\infty} p_{j+n} \sum_{i=0}^j \binom{n+i}{i} a^i \int_0^a x^{j-i} dx \\ &= \frac{1}{n!} \sum_{j=0}^{\infty} p_{j+n} a^{j+1} \sum_{l=0}^j \frac{1}{l+1} (n+j-l) \cdots (1+j-l) \end{aligned}$$

for all $n \geq 1$. Since

$$\sum_{l=0}^j \frac{1}{l+1} \prod_{i=1}^n (i+j-l) = j^n \sum_{l=0}^j \frac{1}{l+1} \prod_{i=1}^n (1+(i-l)j^{-1}) \sim j^n \ln j, \quad j \rightarrow \infty,$$

the statement follows. ■

Proposition 14 Consider a (q, r) -extendable probability generating function f with $f'(r) < \infty$ and the corresponding function (4). We have $\int_0^r |\psi_{q,r}(x)| dx < \infty$, if and only if

$$\sum_{k=2}^{\infty} p_k r^k k \ln k < \infty. \quad (29)$$

If condition (29) does not hold, then the function $\mathcal{L}_{q,r}(x) = \exp \left\{ \int_0^{r-x} \psi_{q,r}(s) ds \right\}$ slowly varies as $x \rightarrow 0$ and $\mathcal{L}_{q,r}(x) \rightarrow 0$.

Proposition 15 Let $f(1) = 1$, $f'(1) = 1$, and $f'''(1) < \infty$ and consider the function (13). We have $\int_0^1 |\psi_{1,1}(x)| dx < \infty$ if and only if

$$\sum_{k=2}^{\infty} p_k k^3 \ln k < \infty. \quad (30)$$

If (30) does not hold, then the function $\mathcal{L}_{1,1}(x) = \exp \left\{ \int_0^{1-x} \psi_{1,1}(s) ds \right\}$ slowly varies as $x \rightarrow 0$ and $\mathcal{L}_{1,1}(x) \rightarrow 0$.

PROOF Propositions 14 and 15 have similar proofs. Here we prove only Proposition 14. Applying Lemma 6, we see that

$$\phi(s) \in [\phi(0), \phi(r)] \subset (0, \infty), \quad s \in [0, r].$$

Thus, in view of $\phi^{(2)}(q, r) < \infty$, we have

$$c_{q,r} := \int_0^r \frac{\phi^{(2)}(q, x) dx}{\phi(x)} < \infty,$$

and it suffices to verify that

$$\int_0^r \phi^{(2)}(r, x) dx < \infty \quad (31)$$

if and only if (29) holds (the integral in (31) may be infinite because $\phi^{(2)}(r, r)$ is allowed to be infinite). Indeed, since

$$\phi^{(2)}(s_1, s_2) = f^{(4)}(q, r, s_1, s_2) = \frac{f^{(3)}(r, s_1, s_2) - f^{(3)}(q, r, s_1)}{s_2 - q},$$

we have

$$\int_{(r+q)/2}^r \phi^{(2)}(r, x) dx = \int_{(r+q)/2}^r \frac{f^{(3)}(r, r, x) - f^{(3)}(q, r, r)}{x - q} dx = \int_{(r+q)/2}^r \frac{f^{(3)}(r, r, x)}{x - q} dx - \frac{\beta}{r - q} \ln 2,$$

implying that (31) is equivalent to $\int_0^r f^{(3)}(r, r, x) dx < \infty$, which in turn is equivalent to (29) by Theorem 13. To finish the proof of Proposition 14, notice that slow variation of $\mathcal{L}_{q,r}(x)$ follows from the representation

$$\mathcal{L}_{q,r}(x) \sim \exp \left\{ c_{q,r} - \gamma \int_x^r \frac{\eta(s) ds}{s} \right\},$$

where

$$\eta(r - s) = \frac{(r - s)\phi^{(2)}(r, s)}{\phi(s)} = \frac{\phi(r) - \phi(s)}{\phi(s)},$$

so that $\eta(x) \rightarrow 0$ as $x \rightarrow 0$, see [2]. ■

Examples. A possibility for $f'(r) < \infty$ and $f''(r) = \infty$ is illustrated by the next example borrowed from [13]. For a given set of four parameters (q, r, a, θ) satisfying $0 \leq q \leq 1 < r < \infty$, $a \in (0, 1)$, $\theta \in (0, 1)$, the function

$$f(s) = r - \{a(r - s)^{-\theta} + (1 - a)(r - q)^{-\theta}\}^{-1/\theta}$$

is a (q, r) -extendable probability generating function. For this example, we have $f'(q) = a$, $f'(r) = a^{-1/\theta}$, $f''(r) = \infty$, and

$$\begin{aligned} f^{(2)}(r, s) &= \{a + (1 - a)(r - s)^\theta (r - q)^{-\theta}\}^{-1/\theta}, \\ f^{(3)}(q, r, s) &= \frac{\{a + (1 - a)(r - s)^\theta (r - q)^{-\theta}\}^{-1/\theta} - 1}{s - q}, \\ f^{(4)}(q, r, s) &= \frac{a^{-1/\theta}(s - q) - (r - q)\{a + (1 - a)(r - s)^\theta (r - q)^{-\theta}\}^{-1/\theta}}{(r - q)(s - q)(r - s)} + \frac{1}{(r - q)(s - q)}. \end{aligned}$$

Since

$$f^{(4)}(q, r, s) \sim \frac{a^{-1/\theta}(1 - a)}{\theta(r - q)^{1+\theta}(r - s)^{1-\theta}}, \quad s \rightarrow r,$$

we conclude that in this case $\int_0^r |\psi_{q,r}(x)| dx < \infty$.

A related example from [13] introduces the case $f'(r) = \infty$, which is not studied here: if $a \in (0, 1)$ and $q \in [0, 1]$, then

$$f(s) = r - (r - q)^{1-a}(r - s)^a$$

is a (q, r) -extendable probability generating function such that $f^{(2)}(r, s) = (\frac{r-q}{r-s})^{1-a}$.

7 Yaglom-type limit theorem

With $f(1) < 1$, a realisation of the branching process has two possible fates: either to be absorbed at the state 0 at a random time T_0 , or to be absorbed at the graveyard state Δ at a random time T_1 . Indeed, by (3), we have

$$\frac{F_t(1) - q}{1 - q} = e^{-\alpha t} \left\{ \frac{r - F_t(1)}{r - 1} \right\}^\gamma \exp \left\{ - \int_{F_t(1)}^1 \psi_{q,r}(x) dx \right\}. \quad (32)$$

Thus, provided $q < 1 < r$, we get

$$P(Z_t = \Delta) = 1 - F_t(1) \rightarrow 1 - q, \quad t \rightarrow \infty.$$

In the defective case, for the overall absorption time

$$T := \min(T_0, T_1) = T_0 \cdot 1_{\{T_0 < \infty, T_1 = \infty\}} + T_1 \cdot 1_{\{T_0 = \infty, T_1 < \infty\}} + \infty \cdot 1_{\{T_0 = \infty, T_1 = \infty\}},$$

we obtain $P(T = \infty) = 0$ and

$$P(T > t) = P(t < T_0 < \infty) + P(t < T_1 < \infty) = F_t(1) - F_t(0),$$

since

$$\begin{aligned} P(t < T_0 < \infty) &= P(Z_t \neq 0, Z_\infty = 0) = P(Z_\infty = 0) - P(Z_t = 0) = q - F_t(0), \\ P(t < T_1 < \infty) &= P(Z_\infty = \Delta) - P(Z_t = \Delta) = 1 - q - (1 - F_t(1)) = F_t(1) - q. \end{aligned}$$

We will establish an asymptotic formula for $P(T > t)$ as $t \rightarrow \infty$, using the following result for $q - F_t(s)$, which is also valid for $f(1) = 1$.

Lemma 16 *In the (q, r) -extendable case, for a given $s \in [0, r)$, we have*

$$F_t(s) = q + K(s)e^{-\alpha t} + \frac{f''(q)}{2\alpha} K^2(s)e^{-2\alpha t} + o(e^{-2\alpha t}), \quad t \rightarrow \infty,$$

where

$$K(s) = (s - q)(r - q)^\gamma (r - s)^{-\gamma} \exp \left\{ \int_s^q \psi_{q,r}(x) dx \right\}.$$

PROOF For $s = q$ the assertion is trivial. By Theorem 3, for $s \in [0, r)$ and $s \neq q$, we have

$$\begin{aligned} F_t(s) - q &= (s - q)e^{-\alpha t} \left(\frac{r - F_t(s)}{r - s} \right)^\gamma \exp \left\{ \int_s^{F_t(s)} \psi_{q,r}(x) dx \right\} \\ &= K(s)e^{-\alpha t} \left(1 - \frac{F_t(s) - q}{r - q} \right)^\gamma \exp \left\{ \int_q^{F_t(s)} \psi_{q,r}(x) dx \right\}. \end{aligned}$$

It remains to observe that as $t \rightarrow \infty$,

$$\begin{aligned} 1 - \left(1 - \frac{F_t(s) - q}{r - q} \right)^\gamma &= (F_t(s) - q) \left(\frac{\gamma}{r - q} + o(1) \right) = \frac{\gamma}{r - q} K(s)e^{-\alpha t} + o(e^{-\alpha t}), \\ \exp \left\{ \int_q^{F_t(s)} \psi_{q,r}(x) dx \right\} - 1 &= (F_t(s) - q) \left(\psi_{q,r}(q) + o(1) \right) \\ &= \frac{\phi^{(2)}(q, q) - \gamma \phi^{(2)}(q, r)}{\phi(q)} \cdot K(s)e^{-\alpha t} + o(e^{-\alpha t}), \end{aligned}$$

and that

$$\frac{\gamma}{r-q} + \frac{\gamma\phi^{(2)}(q,r) - \phi^{(2)}(q,q)}{\phi(q)} = \frac{\gamma}{r-q} + \frac{1-\gamma}{r-q} - \frac{1}{r-q} + \frac{f''(q)}{2\alpha} = \frac{f''(q)}{2\alpha}.$$

■

In the supercritical case, when $0 \leq q < 1 = r$, with probability $1 - q$ the branching process grows exponentially forever without being absorbed at zero or Δ , so that $P(T = \infty) = 1 - q$. This case is excluded in the next asymptotic result.

Theorem 17 *In the (q, r) -extendable case with $r > 1$ and $f'(r) < \infty$, we have*

$$P(T > t) = (K(1) - K(0))e^{-\alpha t} + \frac{f''(q)}{2\alpha}(K^2(1) - K^2(0))e^{-2\alpha t} + o(e^{-2\alpha t}),$$

as $t \rightarrow \infty$, and moreover,

$$P(Z_t = k | T > t) \rightarrow \pi_k, \quad k \geq 1, \quad \sum_{k=1}^{\infty} \pi_k s^k = \frac{K(s) - K(0)}{K(1) - K(0)}.$$

If f is modified linear-fractional, then

$$K(s) = (s - q)(r - q)^{\gamma}(r - s)^{-\gamma},$$

and $\pi_k \sim ck^{1-\gamma}r^{-k}$ as $k \rightarrow \infty$ for some positive constant c .

PROOF Applying Lemma 16 we arrive at the first statement. The stated conditional weak convergence is also a consequence of Lemma 16

$$E(s^{Z_t} | T > t) = \frac{E(s^{Z_t}) - E(s^{Z_t}; T \leq t)}{P(T > t)} = \frac{F_t(s) - F_t(0)}{F_t(1) - F_t(0)} \rightarrow \frac{K(s) - K(0)}{K(1) - K(0)}.$$

Note that

$$K(rs) = (rs - q)(r - q)^{\gamma}r^{-\gamma}(1 - s)^{-\gamma} \exp \left\{ \int_{rs}^q \psi_{q,r}(x) dx \right\} \sim (r - q)^{1+\gamma}r^{-\gamma}(1 - s)^{-\gamma} \mathcal{L}(1 - s),$$

as $s \rightarrow 1$, where

$$\mathcal{L}(1 - s) = \exp \left\{ - \int_q^{rs} \psi_{q,r}(x) dx \right\}$$

is a slowly varying function according to Proposition 14. Therefore, by the Tauberian theorem, we have

$$\sum_{k=1}^n \pi_k r^k \sim n^{\gamma} l_n, \quad n \rightarrow \infty,$$

where l_n is a positive slowly varying sequence.

Turning to a modified linear-fractional f , we use

$$(1 - x)^{-\gamma} = \sum_{k=0}^{\infty} \gamma_k x^k, \quad \gamma_k = \prod_{i=1}^k \left(1 - \frac{1-\gamma}{i} \right),$$

to find

$$(s - q)(1 - s/r)^{-\gamma} = -q + \sum_{k=1}^{\infty} \gamma_{k-1} \left(r - q + \frac{(1-\gamma)q}{k} \right) \frac{s^k}{r^k}.$$

Thus

$$\frac{K(s) - K(0)}{K(1) - K(0)} = \sum_{k=1}^{\infty} c_k \gamma_{k-1} r^{-k} s^k, \quad c_k = \frac{1}{q + (1-q)(1-r^{-1})^{-\gamma}} \left(r - q + \frac{(1-\gamma)q}{k} \right),$$

and it remains to see that $c_k \gamma_{k-1} \sim (r-q)k^{1-\gamma}/c_\gamma$ as $k \rightarrow \infty$, where $c_\gamma = \Gamma(\gamma)(q + (1-q)(1-r^{-1})^{-\gamma})$. ■

Remark. It is interesting to see how two fixed points q and r regulate different aspects of the non-absorption behavior. While the rate of decay of the non-absorption probability is controlled by $\alpha = 1 - f'(q)$, the conditional distribution tails are ruled by the value of r , in that π_k is of order r^{-k} .

Example. In the framework of Proposition 12 we find

$$K(s) = (s-q)(r-q)^\gamma (r-s)^{-\gamma} \left(\frac{p_3 q + w}{p_3 s + w} \right)^{1-\gamma}.$$

Since

$$K(rs) \sim (r-q)^{\gamma+1} r^\gamma \gamma^{1-\gamma} (1-s)^{-\gamma}, \quad s \rightarrow 1,$$

we see that in this case $\sum_{k=1}^n \pi_k r^k \sim cn^\gamma$ as $n \rightarrow \infty$ for some positive c .

8 Limit theorems for the termination time

In this section we consider a family of (q_ϵ, r_ϵ) -extendable branching processes satisfying (6), where, without loss of generality, it is assumed that $\epsilon = 1 - f_\epsilon(1)$. We obtain weak convergence results as $\epsilon \rightarrow 0$ for the termination time $T_{1,\epsilon}$ conditioned on $T_{1,\epsilon} < \infty$.

Lemma 18 *In the nearly subcritical case, when $0 < m_1 < 1$, we have*

$$1 - q_\epsilon \sim \frac{\epsilon}{1 - m_1}, \quad r_\epsilon \rightarrow r \in (1, \infty), \quad \alpha_\epsilon \rightarrow 1 - m_1 \in (0, 1), \quad \beta_\epsilon \rightarrow \beta \in (0, \infty).$$

In the nearly supercritical case, when $m_1 > 1$, we have

$$r_\epsilon - 1 \sim \frac{\epsilon}{m_1 - 1}, \quad q_\epsilon \rightarrow q \in [0, 1), \quad \alpha_\epsilon \rightarrow \alpha \in (0, 1), \quad \beta_\epsilon \rightarrow m_1 - 1 \in (0, \infty).$$

In the nearly critical case, when $m_1 = 1$, we have

$$q_\epsilon \rightarrow 1, \quad r_\epsilon \rightarrow 1, \quad \alpha_\epsilon \rightarrow 0, \quad \beta_\epsilon \rightarrow 0, \quad \gamma_\epsilon \rightarrow 1,$$

and if it is given that

$$d_\epsilon := \frac{1 - q_\epsilon}{r_\epsilon - 1} \rightarrow d \in [0, \infty], \tag{33}$$

then

$$\frac{1 - q_\epsilon}{\sqrt{\epsilon}} \rightarrow \sqrt{\frac{d}{m_2}}, \quad \frac{r_\epsilon - 1}{\sqrt{\epsilon}} \rightarrow \frac{1}{\sqrt{m_2 d}}.$$

PROOF The first two assertions follow from the equalities

$$\epsilon = \phi_\epsilon(1)(1 - q_\epsilon)(r_\epsilon - 1), \quad \phi_\epsilon(q_\epsilon) = \frac{\alpha_\epsilon}{r_\epsilon - q_\epsilon}, \quad \phi_\epsilon(r_\epsilon) = \frac{\beta_\epsilon}{r_\epsilon - q_\epsilon},$$

obtained from Lemma 6. In the nearly critical case, since

$$\phi_\epsilon(q_\epsilon) \rightarrow m_2, \quad \phi_\epsilon(r_\epsilon) \rightarrow m_2, \quad \phi_\epsilon^{(2)}(q_\epsilon, r_\epsilon) \rightarrow m_3,$$

we get

$$m_2(1 - q_\epsilon)(r_\epsilon - 1) \sim \epsilon, \quad \frac{\alpha_\epsilon}{r_\epsilon - q_\epsilon} \rightarrow m_2, \quad \frac{\beta_\epsilon}{r_\epsilon - q_\epsilon} \rightarrow m_2.$$

Using these relations it is easy to verify the statements for the nearly critical case. ■

Theorem 19 *Consider a family of (q_ϵ, r_ϵ) -extendable branching processes satisfying (6) and let $\epsilon \rightarrow 0$.*

(i) *If $m_1 < 1$, then for any fixed $t \geq 0$,*

$$\mathbb{P}(T_{1,\epsilon} \leq t \mid T_{1,\epsilon} < \infty) \rightarrow 1 - e^{(1-m_1)t}.$$

(ii) *If $m_1 > 1$, then for any fixed $u \in (-\infty, \infty)$,*

$$\mathbb{P}(T_{1,\epsilon} \leq \frac{\ln(r_\epsilon - 1)^{-1}}{\beta_\epsilon} + \frac{\ln(1 - q) + u}{m_1 - 1} \mid T_{1,\epsilon} < \infty) \rightarrow \Phi(u),$$

and the limit distribution function satisfies

$$u = \ln \Phi(u) + \int_{1-(1-q)\Phi(u)}^1 \psi(x) dx, \quad (34)$$

where the function

$$\psi(x) = \frac{\phi^{(2)}(1, x)(1 - q)}{\phi(x)} + \frac{m_1 - 1}{(1 - q)(x - q)\phi(x)}$$

takes positive values over $x \in (q, 1]$.

PROOF Put

$$V_\epsilon(t) = \mathbb{P}(T_{1,\epsilon} \leq t \mid T_{1,\epsilon} < \infty)$$

and observe that

$$V_\epsilon(t) = \frac{1 - F_{t,\epsilon}(1)}{1 - q_\epsilon} = 1 - \frac{F_{t,\epsilon}(1) - q_\epsilon}{1 - q_\epsilon}.$$

(i) Referring to (32) we can write an equation for $V_\epsilon(t)$

$$1 - V_\epsilon(t) = e^{-\alpha_\epsilon t} \left[1 + \frac{1 - q_\epsilon}{r_\epsilon - 1} V_\epsilon(t) \right]^{\gamma_\epsilon} \exp \left\{ - \int_{1-(1-q_\epsilon)V_\epsilon(t)}^1 \psi_{q_\epsilon, r_\epsilon}(x) dx \right\}. \quad (35)$$

This and the first part of Lemma 18 imply the assertion in the nearly subcritical case.

(ii) In the nearly supercritical case applying (9) with $s = 1$ we get

$$\beta_\epsilon t + \ln(r_\epsilon - 1) = \ln(r_\epsilon - F_{t,\epsilon}(1)) + \int_{F_{t,\epsilon}(1)}^1 \frac{\phi_\epsilon^{(2)}(r_\epsilon, x) dx}{\phi_\epsilon(x)} + \int_{F_{t,\epsilon}(1)}^1 \frac{\phi(r_\epsilon) dx}{(x - q_\epsilon)\phi_\epsilon(x)}.$$

It follows that the time scaled distribution function $\Phi_\epsilon(u) = V_\epsilon(t_\epsilon(u))$, where

$$t_\epsilon(u) = \frac{\ln(r_\epsilon - 1)^{-1} + \ln(1 - q_\epsilon) + u}{\beta_\epsilon},$$

satisfies

$$u = \ln \left(\frac{r_\epsilon - 1}{1 - q_\epsilon} + \Phi_\epsilon(u) \right) + \int_{1-(1-q_\epsilon)\Phi_\epsilon(u)}^1 \frac{\phi_\epsilon^{(2)}(r_\epsilon, x) dx}{\phi_\epsilon(x)} + \int_{1-(1-q_\epsilon)\Phi_\epsilon(u)}^1 \frac{\phi(r_\epsilon) dx}{(x - q_\epsilon)\phi_\epsilon(x)}.$$

Letting $\epsilon \rightarrow 0$ and using the standard tightness argument based on Helly's selection theorem, we see that $\Phi_\epsilon(u) \rightarrow \Phi(u)$, as the limit distribution is uniquely determined by the equation (34). ■

Theorem 20 Consider a family of (q_ϵ, r_ϵ) -extendable branching processes satisfying (6) with $m_1 = 1$, and assume (33).

(i) If $d = 0$, then for any fixed $t \geq 0$,

$$\mathbb{P}\left(T_{1,\epsilon} \leq \frac{t}{(r_\epsilon - 1)m_2} \mid T_{1,\epsilon} < \infty\right) \rightarrow 1 - e^{-t}.$$

(ii) If $d \in (0, \infty)$, then for any fixed $t \geq 0$,

$$\mathbb{P}\left(T_{1,\epsilon} \leq \frac{t}{a\sqrt{\epsilon}} \mid T_{1,\epsilon} < \infty\right) \rightarrow \frac{e^t - 1}{e^t + d}, \quad a = \sqrt{m_2(d + 1/d)}.$$

(iii) If $d = \infty$, then there is convergence to the standard logistic distribution

$$\mathbb{P}\left(T_{1,\epsilon} \leq \frac{\ln d_\epsilon}{\beta_\epsilon} + \frac{u}{\alpha_\epsilon} \mid T_{1,\epsilon} < \infty\right) \rightarrow \frac{1}{1 + e^{-u}}, \quad u \in (-\infty, \infty).$$

PROOF Items (i) and (ii) are obtained in the same way as Theorem 19 (i) using Lemma 18. To prove (iii) we turn to (35) and find that uniformly over $t \geq 0$,

$$1 - V_\epsilon(t) \sim e^{-\alpha_\epsilon t} \left[1 + d_\epsilon V_\epsilon(t)\right]^{\gamma_\epsilon} \sim e^{-\alpha_\epsilon t} (d_\epsilon)^{\gamma_\epsilon}, \quad \epsilon \rightarrow 0.$$

Choosing here $t = \frac{u + \gamma_\epsilon \ln d_\epsilon}{\alpha_\epsilon}$ we obtain statement (iii). Notice, that condition (6) implies $m_2 \in (0, \infty)$. ■

Example. In the framework of modified linear-fractional generating functions, condition (6) requiring convergence over $s \in [0, s_0]$ with $s_0 > 1$, comes naturally in the form

$$f_\epsilon(s) \rightarrow p_0 + p_1 s + (1 - p_0 - p_1)s^2(1 - p)(1 - ps)^{-1}, \quad s \in [0, 1/p).$$

In this particular case the limit equation in Proposition 19 (ii) simplifies taking the form

$$e^{-u}\Phi(u) = (1 - \Phi(u))^{1/\gamma}.$$

For example, with $\gamma = 1/2$, we get

$$\Phi(u) = 1 - e^{-u}\sqrt{e^u - 1/4}.$$

If $\gamma \in (0, 1)$, then as $u \rightarrow \infty$

$$1 - \Phi(u) \sim e^{-\gamma u}, \quad \Phi(-u) \sim e^{-u}.$$

Remark. Comparing these five asymptotic formulas for conditional distribution of the termination time, we find that the largest typical values are expected in the balanced near critical case with $d = 1$, when $1 - q_\epsilon \sim r_\epsilon - 1$, and

$$\mathbb{P}\left(T_{1,\epsilon} \leq \frac{t}{\sqrt{\epsilon}} \mid T_{1,\epsilon} < \infty\right) \rightarrow \frac{e^{at} - 1}{e^{at} + 1}, \quad a = \sqrt{f''(1)}.$$

If a particle terminates the whole branching process with probability $\epsilon = 10^{-4}$, then in the balanced nearly critical case with $d = 1$ and $m_2 = 1$, this process does not go extinct with approximate probability $\sqrt{\epsilon} = 10^{-2}$. Conditioned on non-extinction, the process will terminate after a time of order 100 seconds (assuming that the average lifelength of a particle is one second).

9 A refined asymptotic formula in the critical case

Proposition 21 Consider a critical branching process satisfying (30). Then for any fixed $s \in [0, 1)$,

$$1 - F_t(s) = \frac{1}{m_2 t} - \frac{m_3 \ln t}{m_2^3 t^2} - \frac{A(s)}{m_2^2 t^2} + o\left(\frac{1}{t^2}\right), \quad t \rightarrow \infty,$$

where

$$A(s) = \frac{1}{1-s} - \frac{m_3}{m_2} \ln(1-s) - m_2 \int_s^1 \psi_{1,1}(x) dx.$$

PROOF By Theorem 7, for a given $s \in [0, 1)$,

$$\frac{1}{1-F_t(s)} - \frac{1}{1-s} = m_2 t + \frac{m_3}{m_2} \ln \frac{1-F_t(s)}{1-s} - m_2 \int_s^{F_t(s)} \psi_{1,1}(x) dx.$$

It follows immediately that $1 - F_t(s) \sim \frac{1}{m_2 t}$ as $t \rightarrow \infty$. Assuming (30) and applying Proposition 15 we obtain from the previous equality

$$\frac{1}{1-F_t(s)} = m_2 t + \frac{m_3}{m_2} \ln(1-F_t(s)) + A(s) + o(1).$$

From here, using relation

$$1 - F_t(s) = \frac{1}{m_2 t} \left(1 - \frac{m_3}{m_2^2} \frac{\ln t}{t} - \frac{A(s) + \epsilon_t(s)}{m_2 t} \right)$$

as the definition of $\epsilon_t(s)$, we find that $\epsilon_t(s) = o(1)$, which is what had to be proven. ■

Plugging in the just proven formula $s = 0$ we obtain the following asymptotic result for the probability of survival by time t .

Corollary 22 *If $f(1) = f'(1) = 1$ and (30) holds, then*

$$P(Z_t > 0) = \frac{1}{m_2 t} - \frac{m_3}{m_2^3} \frac{\ln t}{t^2} - \frac{A(0)}{m_2^2 t^2} + o\left(\frac{1}{t^2}\right), \quad t \rightarrow \infty.$$

Remark. The last asymptotic formula should be compared to a formula on page 248 in [15]:

$$P(Z_t > 0) = \frac{1}{m_2 t} + \frac{m_3}{m_2^3} \frac{\ln t}{t^2} + o\left(\frac{\ln t}{t^2}\right), \quad t \rightarrow \infty.$$

Our formula provides with an expression for a higher order term, and also removes a misprint in Zolotarev's formula affecting the sign of the second term. (For a detailed account on the critical Markov branching processes under weaker moment conditions see [11].)

Corollary 23 *If $f(1) = f'(1) = 1$ and (30) holds, then for any $k \geq 1$,*

$$P(Z_t = k) \sim \frac{h_k}{t^2}, \quad t \rightarrow \infty,$$

with the sequence $\{h_k\}$ being characterised by

$$\sum_{k=1}^{\infty} h_k s^k = \frac{1}{m_2^2} \frac{s}{1-s} + \frac{m_3}{m_2^3} \ln \frac{1}{1-s} + \frac{1}{m_2^2} \int_0^s \psi_{1,1}(x) dx.$$

PROOF The statement follows from

$$\sum_{k=1}^{\infty} P(Z_t = k) s^k = F_t(s) - F_t(0) = \frac{A(s) - A(0)}{m_2^2 t^2} + o\left(\frac{1}{t^2}\right), \quad t \rightarrow \infty.$$

■

Example 1. If f is given by (18), then by Corollary 22,

$$P(Z_t > 0) = \frac{1-p}{p_0 t} - \frac{(1-p)p}{p_0^2} \frac{\ln t}{t^2} - \frac{(1-p)^2}{p_0^2 t^2} + o\left(\frac{1}{t^2}\right),$$

as $t \rightarrow \infty$, and by Corollary 23,

$$t^2 P(Z_t = k) \rightarrow \frac{1-p}{p_0^2} \left(1 - p \frac{k-1}{k}\right), \quad k \geq 1.$$

Example 2. Consider the critical case in the framework of Section 5:

$$f(s) = s + \frac{1}{2}(1-s)^2(2p_3 s + 1 + p_3), \quad p_3 \in [0, 1).$$

By Proposition 21,

$$1 - F_t(s) = \frac{2}{1+3p_3} \frac{1}{t} - \frac{8p_3}{(1+3p_3)^3} \frac{\ln t}{t^2} - \frac{4A(s)}{(1+3p_3)^2 t^2} + o\left(\frac{1}{t^2}\right), \quad t \rightarrow \infty,$$

$$A(s) = \frac{1}{1-s} + \frac{2p_3}{1+3p_3} \ln \frac{2(1+p_3+2p_3 s)}{(1-s)(1+3p_3)},$$

and by Corollary 22,

$$P(Z_t > 0) = \frac{2}{1+3p_3} \frac{1}{t} - \frac{8p_3}{(1+3p_3)^3} \frac{\ln t}{t^2} - \frac{4(1+3p_3) + 8 \ln \frac{2(1+p_3)}{1+3p_3}}{(1+3p_3)^3} \frac{1}{t^2} + o\left(\frac{1}{t^2}\right), \quad t \rightarrow \infty.$$

Corollary 23 holds with

$$\sum_{k=1}^{\infty} h_k s^k = \frac{4}{(1+3p_3)^2} \left(\frac{s}{1-s} + \frac{2p_3}{1+3p_3} \ln \frac{1+p_3+2p_3 s}{(1-s)(1+p_3)} \right).$$

10 A new proof in the supercritical case

Given $f(1) = 1$, the branching process normalised by its mean $M_t = e^{-(m_1-1)t}$ forms a non-negative martingale implying almost sure convergence $Z_t/M_t \rightarrow W$, $t \rightarrow \infty$. Thus for $\rho \geq 0$,

$$E e^{-\rho Z_t / m_t} \rightarrow E e^{-\rho W}, \quad t \rightarrow \infty.$$

In this section we apply the tail generating function technique to give streamlined proofs for classical results concerning the limit Laplace transform $\eta(\rho) = E e^{-\rho W}$.

Proposition 24 *Let $f(1) = 1$ and $m_1 \in (1, \infty)$. If*

$$\sum_{k=2}^{\infty} p_k k \ln k < \infty, \tag{36}$$

then for $\rho > 0$, we have

$$\eta(\rho) = q + (1-q) \left(\frac{1-\eta(\rho)}{\rho} \right)^{\gamma} \exp \left\{ - \int_{\eta(\rho)}^1 \psi_{q,1}(x) dx \right\}, \tag{37}$$

so that $\eta(\rho) \in (q, 1)$. If (36) does not hold, then $\eta(\rho) = 1$, $\rho \geq 0$, so that $P(W = 0) = 1$.

PROOF By Theorem 3 with $r = 1$, in view of $M_t = e^{-\beta t}$, we have

$$\frac{F_t(s) - q}{s - q} = \left\{ M_t \frac{1 - F_t(s)}{1 - s} \right\}^\gamma \exp \left\{ \int_s^{F_t(s)} \psi_{q,1}(x) dx \right\}.$$

Replacing s with $s_t = e^{-\rho/M_t}$ and putting $\eta_t(\rho) = F_t(s_t)$, we obtain

$$\frac{\eta_t(\rho) - q}{s_t - q} \sim \left(\frac{1 - \eta_t(\rho)}{\rho} \right)^\gamma \exp \left\{ \int_{s_t}^{\eta_t(\rho)} \psi_{q,1}(x) dx \right\}, \quad t \rightarrow \infty.$$

Thus, if (36) holds, then due to Proposition 14 with $r = 1$, equation (37) follows, which in turn implies that $\eta(\rho) \in (q, 1)$ for $\rho > 0$.

On the other hand, if (36) does not hold, then again by Proposition 14,

$$\exp \left\{ \int_{s_t}^{\eta_t(\rho)} \psi_{q,1}(x) dx \right\} = \frac{\mathcal{L}_{q,1}(1 - \eta_t(\rho))}{\mathcal{L}_{q,1}(1 - s_t)},$$

where $\mathcal{L}_{q,1}(1 - s_t) \rightarrow 0$ as $t \rightarrow \infty$. We conclude that in this case $\eta_t(\rho) \rightarrow 1$. ■

Corollary 25 *If (36) holds, then $EW = 1$ and there is a positive constant C such that*

$$P(W \leq t | W > 0) \sim Ct^\gamma, \quad t \rightarrow 0.$$

Also, if $m_2 < \infty$, then $EW^2 = \frac{2m_2}{m_1 - 1}$.

PROOF In view of

$$EW^n = (-1)^n \eta^{(n)}(0), \quad n \geq 1,$$

equation (37) implies $EW = 1$. Furthermore, by a Taylor expansion we have

$$\frac{1 - \eta(\rho)}{\rho} = 1 - \frac{\eta''(0)}{2} \rho + o(\rho),$$

which together with (37) and (5) give

$$\frac{\gamma \eta''(0)}{2} = -\frac{1}{1 - q} - \psi_{q,1}(1) = \frac{\gamma m_2}{\beta}.$$

Thus, we obtain $EW^2 = \frac{2m_2}{\beta}$. Next, observe that as $\rho \rightarrow \infty$, equation (37) gives

$$\rho^\gamma (\eta(\rho) - q) \rightarrow (1 - q) q^\gamma \exp \left\{ - \int_q^1 \psi_{q,1}(x) dx \right\},$$

implying

$$E(e^{-\rho W} | W > 0) = \frac{\eta(\rho) - q}{1 - q} \sim C_1 \rho^{-\gamma}, \quad \rho \rightarrow \infty,$$

for some $C_1 \in (0, \infty)$. By the Tauberian Theorem 2 from [3, Ch. XIII.5] this brings the second claim of the corollary. ■

Examples. For a modified linear-fractional f , equation (37) can be written as

$$\eta(\rho) = q + (1 - q) \left(\frac{1 - \eta(\rho)}{\rho} \right)^\gamma.$$

In particular, if $\gamma = 1$, the limit distribution both is exponential, in that

$$E(e^{-\rho W} | W > 0) = \frac{1 - q}{1 - q + \rho}.$$

If $\gamma = \frac{1}{2}$, then

$$\eta(\rho) = q + (1 - q) \frac{\sqrt{(1 - q)(1 - q + 4\rho)} - 1 + q}{2\rho}.$$

For the example from Section 5, as compared to the previous example, we obtain an extra term in the equation

$$\eta(\rho) = q + (1 - q) \left(\frac{1 - \eta(\rho)}{\rho} \right)^\gamma \left(\frac{1 + p_3 q r + (q + r) p_3 \eta(\rho)}{1 + p_3 q + p_3 r + p_3 q r} \right)^{1 - \gamma}.$$

Remark. Equation (37) should be compared to its counterpart stated in Theorem 3 from [1, Ch III.8], which can be rewritten as

$$\eta(\rho) = 1 - \rho \exp \left\{ \int_{\eta(\rho)}^1 \left(\frac{m_1 - 1}{f(x) - x} + \frac{1}{1 - x} \right) dx \right\}. \quad (38)$$

To demonstrate equivalence of these two equations we use the chain of equalities

$$\begin{aligned} \frac{m_1 - 1}{f(x) - x} + \frac{1}{1 - x} &= \frac{1}{(1 - x)(q - x)\phi(x)} (f^{(2)}(1, 1) - f^{(2)}(1, x)) = \frac{f^{(3)}(1, 1, x)}{(q - x)\phi(x)} \\ &= \frac{f^{(3)}(q, 1, 1)}{(q - x)\phi(x)} - \frac{f^{(4)}(q, 1, 1, x)}{\phi(x)} = \frac{\phi(1)}{(q - x)\phi(q)} + \frac{\phi(1)\phi^{(2)}(q, x)}{\phi(q)\phi(x)} - \frac{\phi^{(2)}(1, x)}{\phi(x)}. \end{aligned}$$

Since $\frac{\phi(1)}{\phi(q)} = \gamma^{-1}$, we arrive at (37) after observing that according to (38)

$$\gamma \ln \frac{1 - \eta(\rho)}{\rho} = \gamma \int_{\eta(\rho)}^1 \left(\frac{m_1 - 1}{f(x) - x} + \frac{1}{1 - x} \right) dx = \ln \frac{\eta(\rho) - q}{1 - q} + \int_{\eta(\rho)}^1 \psi_{q,1}(x) dx.$$

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